

On the theory of relaxation in nonlinear elasticity with constraints on the determinant

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We consider vectorial variational problems in nonlinear elasticity of the form $I[u] = \int W(Du)dx$, where W is continuous on matrices with positive determinant and diverges to infinity along sequences of matrices whose determinant is positive and tends to zero. We show that, under suitable growth assumptions, the functional $\int W^{qc}(Du)dx$ is an upper bound on the relaxation of I , and coincides with the relaxation if the quasiconvex envelope W^{qc} of W is polyconvex and has p -growth from below with $p \geq n$. This includes several physically relevant examples. We also show how a constraint of incompressibility can be incorporated in our results.

1 Introduction

Starting with the work of Morrey [32], the concept of quasiconvexity has been fundamental in the study of the relaxation of vectorial problems in the calculus of variations, see for example [18, 34, 39]. In particular, if $W : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is continuous and has p -growth then the relaxation of the functional $I : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R}$,

$$I[u] = \int_{\Omega} W(Du)dx \quad (1.1)$$

is given by

$$I^*[u] = \int_{\Omega} W^{qc}(Du)dx, \quad (1.2)$$

where W^{qc} is the quasiconvex envelope of W , see [32, 1, 18]. Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz set, W^{qc} is defined by (1.3) below. The computation of W^{qc} is in general difficult, but it was performed in a number of special cases with high symmetry, see for example [20, 17, 41, 16]. The key strategy is to construct specific test functions using lamination and rank-one convexity to prove an upper bound, and then to show that the resulting expression is polyconvex, which delivers the lower bound.

One of the main applications of the vectorial calculus of variations is nonlinear elasticity with $m = n$. The physical constraint of non-interpenetration

of matter leads naturally to the requirement of injectivity of the deformation u , a complex nonlocal condition which is often replaced by the simpler condition that $\det Du > 0$ almost everywhere. Correspondingly, one assumes that the energy density W diverges when the determinant of the argument is positive and tends to zero. Such energy densities are not continuous on $\mathbb{R}^{n \times n}$ and do not have p -growth from above for any p , hence the general relaxation theorem is not applicable. Starting with the work of Ball [2] a large body of work developed with the aim of proving existence of minimizers for variational problems with the constraint $\det Du > 0$, mainly building upon the concept of polyconvexity. In contrast, to the best of our knowledge, there is no physically-relevant functional incorporating the nonlinear constraint $\det Du > 0$ for which a nontrivial relaxation is known.

The significance of the constraint $\det Du > 0$ depends dramatically on the growth exponent p of the energy, and two main regimes emerge. If $p \geq n$ the deformation u is necessarily continuous. This follows from the Sobolev embedding theorem for $p > n$ and in the case $p = n$ from the work of Vodop'janov and Gol'dšteĭn [43], see also [42]. Further, if a sequence u_j converges weakly to some u in $W^{1,n}$ and $\det Du_j > 0$ almost everywhere, then necessarily $\det Du \geq 0$ almost everywhere. For $p > n$ this follows directly from the properties of the determinant, which in particular give $\det Du_j \rightarrow \det Du$ in L^1 [2, Cor. 6.2.2], and still holds for $p = n$, see [8, Th. 4.1(ii)]. A related treatment with the distributional determinant instead of the pointwise determinant is still possible if $p > n - 1$ and a generalized invertibility condition is used instead of $\det Du > 0$ [36], see also [35, 14, 23, 25] for subsequent developments.

The case $p < n$ is substantially different, since the deformations are not continuous and can develop holes, as was first shown by Ball [4]. Correspondingly, the constraint of having positive determinant does not pass to the limit and the relaxed problem has a substantially different structure, see for example [8, 28, 27] for further developments. Relaxation in a related situation in which the constraints are lost after rank-one convexification was discussed in [10]. We shall not discuss these cases further here.

The proof of the classical relaxation theorem for continuous integrands is based on a truncation procedure in which one replaces a sequence $u_j \in W^{1,p}$ which converges weakly to an affine function by a sequence with the same weak limit, the same energy, and which is affine on the boundary. It is currently unknown if a similar construction can be done if a constraint on the determinant has to be preserved. In two spatial dimensions and under the assumption that u is bilipschitz a solution was given in [11] building upon an involved construction of bilipschitz extensions by Daneri and Pratelli [19]. The approximation of Hölder-continuous homeomorphisms was obtained in [9]. The situation with Sobolev functions is substantially more complex, and was up to now only solved in the case $p = n = 2$, see [26]. A related problematic arises in the relaxation of problems with mixed growth, see for example [29].

In this paper we prove that the functional (1.2) gives an upper bound on the relaxation of (1.1) for a class of energy densities which are infinite on matrices F with $\det F \leq 0$ and have p -growth for some $p \geq 1$ on the set $\{\det F > 0\}$, see Theorem 2.1 below. If W^{qc} is polyconvex and $p \geq n$ then (1.2) coincides with the relaxation of (1.1), see Theorem 2.2 below. The growth assumption can be somewhat relaxed if a suitable integrability of the cofactor is assumed, see Remark 2.3. Since all known explicit quasiconvex envelopes W^{qc} are quasiconvex, our result fully characterizes the relaxation in all cases where W^{qc} has been computed and the pointwise determinant constraint survives the relaxation. We also show that our results can be generalized to problems where $\det Du = 1$ almost everywhere, see Section 2.2 below.

Notation: We denote by $\mathbb{R}_+^{n \times n} = \{F \in \mathbb{R}^{n \times n} : \det F > 0\}$ the set of orientation-preserving matrices, by $B(r, x_0)$ the open ball in \mathbb{R}^n and set $B_r = B(r, 0)$. Finally $\int_E f dx$ denotes the mean value of f on E .

We define the quasiconvex envelope $W^{\text{qc}} : \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ of a Borel-measurable function $W : \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ by

$$W^{\text{qc}}(F) = \inf \left\{ \int_{B_1} W(D\varphi) dx : \varphi \in W^{1,\infty}(B_1; \mathbb{R}^n), \varphi(x) = Fx \text{ for } x \in \partial B_1 \right\}. \quad (1.3)$$

This is not necessarily the same as the largest finite-valued quasiconvex function below W , see Remark 3.5 below.

We say that a function $f : \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ is polyconvex if there is a lower semicontinuous and convex function $g : \mathbb{R}^{\tau(n)} \rightarrow [0, \infty]$ such that $f(F) = g(M(F))$, where $M(F)$ denotes all minors of F [2, 18, 34]. In particular, if $n = 2$ then $M(F) = (F, \det F)$, if $n = 3$ then $M(F) = (F, \text{cof } F, \det F)$. The requirement of lower semicontinuity of g is often not included in the definition but instead enforced through appropriate growth conditions. For finite-valued functions this makes no difference, but for extended-valued we believe the present one to be the definition more naturally related to lower semicontinuity, as the example $g(\det F) = 0$ if $\det F > 0$, $g(\det F) = \infty$ otherwise, with the sequence $u(x) = x/j$ shows, see also the discussion in [31].

2 Main results

2.1 Relaxation of orientation-preserving models

The main result of the paper is a relaxation theorem for coercive variational problems in nonlinear elasticity incorporating a constraint on the determinant, see Theorem 2.2 below. Our key new contribution is a construction for the upper bound which preserves the positive-determinant constraint and leads to the following statement.

Theorem 2.1. Let $W \in C^0(\mathbb{R}_+^{n \times n}, [0, \infty))$ obey

$$\frac{1}{c}|F|^p + \frac{1}{c}\theta(\det F) - c \leq W(F) \leq c|F|^p + c\theta(\det F) + c \quad (2.1)$$

for some $p \geq 1$ and $c > 0$, where $\theta : (0, \infty) \rightarrow [0, \infty)$ is convex and satisfies

$$\theta(xy) \leq c(1 + \theta(x))(1 + \theta(y)) \text{ for all } x, y \in (0, \infty), \quad (2.2)$$

and extend W to $\mathbb{R}^{n \times n}$ by $W(F) = \infty$ if $\det F \leq 0$. Let W^{qc} be defined as in (1.3), $\Omega \subset \mathbb{R}^n$ open, bounded and Lipschitz. For any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ there is a sequence $u_j \in W^{1,p}(\Omega; \mathbb{R}^n)$ which converges weakly to u such that $u_j - u \in W_0^{1,p}$ for all j and

$$\limsup_{j \rightarrow \infty} \int_{\Omega} W(Du_j) dx \leq \int_{\Omega} W^{\text{qc}}(Du) dx.$$

Proof. If $\det Du > 0$ almost everywhere the statement follows from Lemma 3.2 in Section 3 below. From the definition one immediately obtains $W^{\text{qc}}(F) = \infty$ if $\det F \leq 0$, therefore in the other case a constant sequence will do. \square

If the coercivity exponent p is at least n , then the determinant is an L^1 function and weakly continuous in compact subsets [33], therefore the constraint on the determinant passes to the limit. Complementing Theorem 2.1 with existing compactness and lower semicontinuity results, based on the concept of polyconvexity [2], leads to a full relaxation and existence statement.

Theorem 2.2. Let $W \in C^0(\mathbb{R}_+^{n \times n}, [0, \infty))$ obey (2.1–2.2) with $p \geq n$ and

$$\lim_{t \rightarrow 0} \theta(t) = \infty,$$

and extend W by $W(F) = \infty$ to the set $\{\det F \leq 0\}$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz, connected set,

$$X = \{u \in W^{1,p}(\Omega; \mathbb{R}^n) : \det Du > 0 \text{ a.e.}\},$$

and $f \in C^0(\mathbb{R}^n)$ with $|f(t)| \leq c(1 + |t|^q)$ for some $q \in [0, p)$. We define W^{qc} as in (1.3) and the functionals $E, E^* : L^1(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$E[u] = \int_{\Omega} (W(Du) + f(u)) dx \text{ and } E^*[u] = \int_{\Omega} (W^{\text{qc}}(Du) + f(u)) dx$$

for $u \in X$, and $E = E^* = \infty$ on $L^1 \setminus X$. Finally assume that $W^{\text{qc}} = W^{\text{pc}}$. Then the following assertions hold:

- (i) E^* is the relaxation of E with respect to strong L^1 convergence, in the sense that

$$E^*[u] = \inf \{ \liminf_{j \rightarrow \infty} E[u_j] : u_j \in L^1(\Omega; \mathbb{R}^n), u_j \rightarrow u \text{ in } L^1 \}.$$

(ii) The same holds if, for any given relatively open set $\Gamma_D \subset \partial\Omega$ and $u_0 \in X$, the functionals E and E^* are set to be ∞ outside

$$\tilde{X} = X \cap \{u = u_0 \text{ on } \Gamma_D\}.$$

(iii) The functional E^* has a minimizer in the space \tilde{X} .

Proof. By Theorem 2.1 for any $u \in X$ there is $(u_j)_{j \in \mathbb{N}} \subset X$ with $u_j = u$ on $\partial\Omega$, $u_j \rightarrow u$ for $j \rightarrow \infty$ in L^p , and $\limsup_{j \rightarrow \infty} E[u_j] \leq E^*[u]$. This proves the upper bound in both cases.

Let now $(u_j)_{j \in \mathbb{N}}$ be a sequence in X with $E^*[u_j] \leq C < \infty$ for all j . From (2.1) one immediately obtains $W^{\text{qc}}(F) \geq |F|^p/c - c$. The growth condition on f ensures, since $q < p$, that $\int_{\Omega} W^{\text{qc}}(Du) dx \leq C' < \infty$ for all j . Taking a subsequence we can assume $u_j \rightarrow u$ in $W^{1,p}$ for some $u \in W^{1,p}(\Omega; \mathbb{R}^n)$. By the continuity of the trace, if $u_j = u_0$ on Γ_D for all j then $u = u_0$ on Γ_D .

In order to show that $u \in X$ it only remains to prove the condition on the determinant. By [33] we have $\det Du_j \rightarrow \det Du$ in $L^1(K)$ for all $K \subset\subset \Omega$. Since θ diverges at 0 the weak limit is positive almost everywhere and $u \in X$.

To prove lower semicontinuity we let $W^{\text{qc}}(F) = g(M(F))$, with g convex and lower semicontinuous, and fix a compact set $K \subset \Omega$. As discussed above, we have $\det Du_j \rightarrow \det Du$ in $L^1(K)$ for all $K \subset\subset \Omega$. The other minors converge also weakly in L^1 , since $u_j \rightarrow u$ in $W^{1,p}$ with $p > n - 1$. Therefore $M(Du_j) \rightarrow M(Du)$ in $L^1(K)$ and using Jensen's inequality and the convexity of g we obtain

$$\begin{aligned} \int_K W^{\text{qc}}(Du) dx &= \int_K g(M(Du)) dx \leq \liminf_{j \rightarrow \infty} \int_K g(M(Du_j)) dx \\ &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} W^{\text{qc}}(Du_j) dx. \end{aligned}$$

The term $\int_{\Omega} f(u) dx$ is continuous. Taking the supremum over all compact subsets $K \subset \Omega$ gives $E^*[u] \leq \liminf E^*[u_j] \leq \liminf E[u_j]$ and concludes the proof. \square

Remark 2.3. (i) The result of Theorem 2.1 can be extended to functions W which obey

$$W(FG) \leq c(1 + W(F))(1 + W(G)) \quad (2.3)$$

and

$$\frac{1}{c}|F| - c \leq W(F) \leq cW^{\text{qc}}(F) + c \quad (2.4)$$

instead of (2.1) and (2.2). We discuss in Appendix B the required modifications to the proof.

- (ii) Using this generalization, one can extend Theorem 2.2 to a situation in which W obeys (2.3), (2.4) and the growth condition

$$\frac{1}{c}|F|^p + \frac{1}{c}|\operatorname{cof} F|^q + \frac{1}{c}\theta(\det F) - c \leq W(F),$$

corresponding to the spaces $\mathcal{A}_{p,q}$ introduced by Ball [2], see also [42].

- (iii) A different picture arises if one instead uses a constraint on the pointwise determinant, with the material becoming substantially softer, see [4, 8] and [28, 27, 38].

2.2 Relaxation of incompressible models

We deal with integrands which are defined on the set of volume-preserving matrices $\Sigma = \{F \in \mathbb{R}^{n \times n} : \det F = 1\}$ and which have p -growth. In this framework, we prove the following result.

Theorem 2.4. *Let $W \in C^0(\Sigma, [0, \infty))$ obey*

$$\frac{1}{c}|F|^p - c \leq W(F) \leq c|F|^p + c \quad (2.5)$$

for some $p \geq 1$ and $c > 0$, and set $W(F) = \infty$ if $\det F \neq 1$. Let W^{qc} be defined as in (1.3), $\Omega \subset \mathbb{R}^n$ open, bounded and Lipschitz. For any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ there is a sequence $u_j \in W^{1,p}(\Omega; \mathbb{R}^n)$ which converges weakly to u such that $u_j - u \in W_0^{1,p}$ and

$$\limsup_{j \rightarrow \infty} \int_{\Omega} W(Du_j) dx \leq \int_{\Omega} W^{\text{qc}}(Du_j) dx.$$

Proof. The statement follows from Lemma 4.3 below. \square

Also in this case, if coercivity is sufficient a full relaxation statement follows.

Theorem 2.5. *Let $W \in C^0(\Sigma, [0, \infty))$ obey (2.5) for some $p \geq n$, let $\Omega \subset \mathbb{R}^n$ be open bounded, Lipschitz, connected,*

$$X = \{u \in W^{1,p}(\Omega; \mathbb{R}^n) : \det Du = 1 \text{ a.e.}\}.$$

We set $W(F) = \infty$ if $\det F \neq 1$, define W^{qc} as in (1.3) and, for $f \in C^0(\mathbb{R}^n)$ with $|f(t)| \leq c(1 + |t|^q)$ for some $q < p$,

$$E[u] = \int_{\Omega} (W(Du) + f(u)) dx \text{ and } E^*[u] = \int_{\Omega} (W^{\text{qc}}(Du) + f(u)) dx$$

for $u \in X$, and $E = E^ = \infty$ on $L^1 \setminus X$. Finally assume that $W^{\text{qc}} = W^{\text{pc}}$. Then the following assertions hold.*

(i) E^* is the relaxation of E with respect to strong L^1 convergence, in the sense that

$$E^*[u] = \inf \{ \liminf_{j \rightarrow \infty} E[u_j] : u_j \in L^1(\Omega; \mathbb{R}^n), u_j \rightarrow u \text{ in } L^1 \}.$$

(ii) The same holds if, for any given relatively open set $\Gamma_D \subset \partial\Omega$ and $u_0 \in X$, the functionals E and E^* are set to be ∞ outside

$$\tilde{X} = X \cap \{u = u_0 \text{ on } \Gamma_D\}.$$

(iii) The functional E^* has a minimizer in the space \tilde{X} .

Proof. The proof is analogous to the proof of Theorem 2.2. \square

2.3 Examples

We first consider the two-well problem in two dimensions, a classical model for microstructure in martensite [6, 7, 12, 5]. Precisely, we define

$$W_{2W}(F) = \text{dist}^2(F, SO(2)U_1 \cup SO(2)U_2) + \theta(\det F),$$

where

$$U_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

for some fixed $\lambda > 1$ are the eigenstrains of the two martensitic phases and $\theta \in C^0((0, \infty), [0, \infty))$ is a convex function which obeys (2.2) and with $\lim_{t \rightarrow 0} \theta(t) = \infty$, for example $f(t) = (t - 1/t)^2$, extended with $\theta = \infty$ on $(-\infty, 0]$. One is then interested in the functional $I_{2W} : W^{1,2}(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$,

$$I_{2W}[u] = \int_{\Omega} W_{2W}(Du) \, dx.$$

In [16] it was shown that the quasiconvex envelope of W_{2W} is given by

$$W_{2W}^{\text{qc}}(F) = h(|Fv|, |Fw|, \det(F)) + \theta(\det F),$$

where $v = (e_1 + e_1)/\sqrt{2}$, $w = (e_1 - e_1)/\sqrt{2}$ and h is defined by

$$h(x, y, d) = \min_{\xi \in [x, \infty), \eta \in [y, \infty)} \left(\xi^2 + \eta^2 + |U_1|^2 - 2\sqrt{A(\xi, \eta, d)} \right),$$

with

$$A(x, y, d) = (x^2 + y^2) \frac{|U_1|^2}{2} + \left(\lambda^2 - \frac{1}{\lambda^2} \right) \sqrt{x^2 y^2 - d^2} + 2d,$$

and that W_{2W}^{qc} is polyconvex. Theorem 2.2 then shows that the relaxation of I_{2W} is given by

$$I_{2W}^*[u] = \int_{\Omega} W_{2W}^{\text{qc}}(Du) dx.$$

A corresponding result holds for models with one potential well only, which can be recovered setting $\lambda = 1$ in the previous expressions, the quasiconvex envelope is given for example in [40].

A related situation with the incompressibility constraint can be obtained from the study of nematic elastomers [21, 40, 20, 44, 41]. They are composite materials in which a rubber (polymer) matrix is coupled to a nematic liquid crystal. The rubber has entropic elasticity and is usually modeled as incompressible; the ordering of the nematic liquid crystal leads to elongation in the direction of the nematic order parameter. After minimizing out the nematic director, the standard model [44] can be cast in the form

$$W_{\text{ne}}(F) = \begin{cases} \sum_{i=1}^n \left(\frac{\lambda_i(F)}{\gamma_i} \right)^p & \text{if } \det F = 1 \\ \infty & \text{if } \det F \neq 1 \end{cases}$$

where $\lambda_i(F)$ are the singular values of F , i.e., the eigenvalues of $(F^T F)^{1/2}$, and $\gamma_i \in (0, \infty)$ are material parameters with $\prod_{i=1}^n \gamma_i = 1$.

In two dimensions one can assume $\gamma_2 = 1/\gamma_1 > 1$ and, taking the natural exponent $p = 2$, one can show that [21]

$$W_{\text{ne}}^{\text{qc}}(F) = \begin{cases} \infty & \text{if } \det F \neq 1 \\ 2 & \text{if } \lambda_2(F) \leq \gamma_2 \text{ and } \det F = 1 \\ W_{\text{ne}}(F) & \text{otherwise.} \end{cases}$$

Further, the function $W_{\text{ne}}^{\text{qc}}$ is polyconvex [21]. Therefore Theorem 2.4 shows that the relaxation of I_{ne} is given by I_{ne}^* .

In the physically relevant situation $n = 3$ a similar expression is analytically known and turns out to be polyconvex for all $p \geq 1$ [20]. Theorem 2.2 then states that I_{ne}^* is the relaxation of I_{ne} for all $p \geq 3$. This does not include, however, the physically relevant exponent $p = 2$. In this case, as noted in the introduction, the variational model with the pointwise constraint on the determinant would predict cavitation under tension. Tensile experiments are however carried out in a regime in which cavitation does not occur, possibly due to metastability or to additional energy contributions not modeled by W_{ne} [44]. Indeed, numerical simulations based on the quasiconvex envelope $W_{\text{ne}}^{\text{qc}}$ and excluding cavitation via the choice of a the finite-element space of continuous function lead to good agreement with experimental observations [15].

2.4 Strategy of the proof

The “classical” construction of the upper bound for relaxation theorems is based on density and interpolation. Given a function $u \in W^{1,p}$ one first approximates it strongly in $W^{1,p}$ by a piecewise affine function and then uses a “good” test function φ in each of the pieces where u is affine. In doing this it is important that the energy is continuous along the approximating sequence. In our case the energy is infinite on all deformations which are not orientation-preserving, therefore one would need to approximate orientation-preserving $W^{1,p}$ maps with piecewise affine orientation-preserving maps, a problem which, as we discussed above, is very difficult and in general unsolved.

Here we show that the passage through piecewise affine maps is not needed. The key idea is to locally use the *composition* of the limiting map u with the map with oscillating gradient φ . Taking the composition preserves the conditions on the determinant, and the map $u \circ \varphi$ belongs to $W^{1,p}$ since $u \in W^{1,p}$ and φ is Lipschitz. The key point is to prove that the unrelaxed energy of the composition is close to the relaxed energy of u .

For simplicity we focus here on a small ball B where Du is close to the identity (after a change of variables this is true around any Lebesgue point of Du). Let $\varphi \in W^{1,\infty}(B, \mathbb{R}^n)$ be a map from (1.3), which in particular is the identity on the boundary of B . The key idea is to define $z = u \circ \varphi$, so that $Dz = Du \circ \varphi D\varphi$. If Du is close to the identity and Dz is close to $D\varphi$ one then (heuristically) obtains

$$\int_B W(Dz) dx \sim \int_B W(D\varphi) dx \sim |B| W^{\text{qc}}(\text{Id}) dx \sim \int_B W^{\text{qc}}(Du) dx.$$

This estimate can be made precise if Du is uniformly close to the identity and W is uniformly continuous. Both properties are true only in appropriate subsets, and hence the main part of this work is devoted to the treatment of the exceptional sets, under the assumption that $\|Du - \text{Id}\|_{L^p(B)}$ is small.

In the incompressible case, since φ is a volume-preserving bilipschitz map a change of variables shows that the smallness of $Du - \text{Id}$ in L^p immediately translates into the smallness of $(Du - \text{Id}) \circ \varphi$ in L^p and hence of the contribution of the set where $Dz - D\varphi$ is large. This renders the volume-preserving construction in Lemma 4.2 simpler than the corresponding one in the orientation-preserving case.

In the orientation-preserving case the situation is more complex, since a factor $\det D\varphi$ arises from the change of variables formula. One of the problematic terms is

$$\int_{\{|Du - \text{Id}| \circ \varphi > \varepsilon\}} |Du - \text{Id}|^p(\varphi(x)) dx = \int_{\{|Du - \text{Id}| > \varepsilon\}} \frac{|Du - \text{Id}|^p(y)}{\det D\varphi(y)} dy.$$

The fact that φ is Lipschitz provides a bound on $\det D\varphi$ but not on its inverse. However, by the very same change of variables formula we know that

$1/\det D\varphi \in L^1$, therefore the above expression has the form of the integral of the product of two L^1 functions. This is in general not defined, but can be controlled if one of the two functions is first shifted by an “appropriate amount”, as done for example when taking the convolution of two L^1 functions. Precisely, this means that for every choice of $f, g \in L^1(B_1)$ there are many $a_0 \in B_{1/2}$ such that

$$\int_{B_{1/2}} f(x)g(x - a_0)dx \leq \|f\|_{L^1(B_1)}\|g\|_{L^1(B_1)},$$

see Lemma 3.1 below for details.

In both cases the local construction is then extended to a global one by a covering argument, see Lemma 3.3 below.

3 Construction of orientation-preserving maps

3.1 Local construction

Before presenting the construction we show how the translation is exploited. We focus here on the derivation of the estimates, and address measurability and weak differentiability issues in Appendix A.

Lemma 3.1. *Let $\psi \in W^{1,\infty}(B_r; \overline{B_r})$, $g \in L^1(B_r)$, $f \in L^1(B(x_0, 2r))$ for some $x_0 \in \mathbb{R}^n$, $r > 0$. Then there exists a measurable set $E \subset B(x_0, r)$ of positive \mathcal{L}^n measure with the following property. For $a_0 \in E$ the function*

$$\tilde{f}(x) = f(\psi(x - a_0) + a_0)g(x - a_0)$$

belongs to $L^1(B(a_0, r))$ with

$$\|\tilde{f}\|_{L^1(B(a_0, r))} \leq \frac{1}{|B_r|} \|f\|_{L^1(B(x_0, 2r))} \|g\|_{L^1(B_r)}.$$

Proof. We can assume without loss of generality that $f, g \geq 0$. The function $(x, a_0) \mapsto \tilde{f}(x)$ is \mathcal{L}^{2n} -measurable by Lemma A.1. We define $h : B(x_0, r) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$h(a_0) = \int_{B(a_0, r)} \tilde{f}(x) dx = \int_{B(a_0, r)} f(\psi(x - a_0) + a_0)g(x - a_0) dx$$

and change variables to

$$h(a_0) = \int_{B_r} f(\psi(x') + a_0)g(x')dx'.$$

We integrate over all $a_0 \in B(x_0, r)$ and interchange the order of integration to obtain

$$\begin{aligned} \int_{B(x_0, r)} h(a_0) da_0 &= \int_{B_r} \left(\int_{B(x_0, r)} f(\psi(x') + a_0) g(x') da_0 \right) dx' \\ &\leq \|f\|_{L^1(B(x_0, 2r))} \|g\|_{L^1(B_r)}. \end{aligned}$$

To conclude we observe that h cannot be almost everywhere larger than its average. \square

Lemma 3.2. *Assume that the function $W \in C^0(\mathbb{R}_+^{n \times n}, [0, \infty))$ satisfies the growth condition (2.1) with $p \geq 1$ and the structure condition (2.2) and fix $F \in \mathbb{R}_+^{n \times n}$ and $\eta > 0$. Then there is $\delta > 0$ such that for any $B = B(x_0, r)$ and $u \in W^{1,p}(B, \mathbb{R}^n)$ with*

$$\int_B (|Du - F|^p + |\theta(\det Du) - \theta(\det F)|) dx \leq \delta \text{ and } \det Du > 0 \text{ a.e.} \quad (3.1)$$

there are $a_0 \in B(x_0, r/2)$ and $z \in W^{1,p}(B, \mathbb{R}^n)$ with $\det Dz > 0$ a.e., $z = u$ on $B(x_0, r) \setminus B(a_0, r/2)$ and

$$\int_{B(a_0, r/2)} W(Dz) dx \leq \int_{B(a_0, r/2)} (W^{\text{qc}}(Du) + \eta) dx. \quad (3.2)$$

Additionally,

$$\int_B |u - z|^p dx \leq cr^p \int_B (W^{\text{qc}}(Du) + 1) dx.$$

If u is Lipschitz, then the same is true for z .

Proof. The L^p bound follows from the bound on $W(Dz)$ using Poincaré and the growth condition, hence we only need to prove (3.2).

By the definition of $W^{\text{qc}}(F)$ there is $\varphi_\eta \in W^{1,\infty}(B_{r/2}, \mathbb{R}^n)$ such that $\varphi_\eta(x) = Fx$ on $\partial B_{r/2}$ and

$$\int_{B_{r/2}} W(D\varphi_\eta) dx \leq W^{\text{qc}}(F) + \eta. \quad (3.3)$$

By the growth condition (2.1) we have $\theta(\det D\varphi_\eta) \in L^1(B_{r/2})$, and with (2.2) also $\theta(\det(F^{-1}D\varphi_\eta)) \in L^1(B_{r/2})$. Since $\det D\varphi_\eta > 0$ almost everywhere there is $\gamma > 0$ (depending on F and η) such that

$$\begin{aligned} &\int_{B_{r/2} \cap \{\det D\varphi_\eta < \gamma\}} (1 + \theta(\det(F^{-1}D\varphi_\eta))) dx \\ &\leq \frac{|B_{r/2}| \eta}{(1 + \|F^{-1}D\varphi_\eta\|_{L^\infty}^p)(1 + |F|^p + \theta(\det F))}. \end{aligned} \quad (3.4)$$

The choice of the constant on the right-hand side will become clear after (3.11).

The function $F^{-1}\varphi_\eta$ is Lipschitz continuous and therefore, by [3, Theorem 1], $F^{-1}\varphi_\eta(B_{r/2}) \subset \overline{B_{r/2}}$. For some $a_0 \in B(x_0, r/2)$ chosen below, we construct the function $z : B = B(x_0, r) \rightarrow \mathbb{R}^n$ by

$$z(x) = \begin{cases} u(F^{-1}\varphi_\eta(x - a_0) + a_0) & \text{if } x \in B' = B(a_0, r/2), \\ u(x) & \text{otherwise.} \end{cases}$$

By Lemma A.2 (with $\psi = F^{-1}\varphi_\eta$), there exists a null set N such for all choices of $a_0 \notin N$ the first expression belongs to $W^{1,1}$. Further we can compute its weak derivative by the usual chain rule, and the traces on $\partial B'$ of the two expressions coincide. In particular, $z \in W^{1,1}(B'; \mathbb{R}^n)$. In order to obtain an estimate on the derivative we choose $a_0 \in E \setminus N$ via Lemma 3.1, applied to the ball B with $f = |Du - F|^p + |\theta(\det Du) - \theta(\det F)|$ and $g = 1 + \theta(\det(F^{-1}D\varphi_\eta))$. Then

$$\int_{B'} (1 + \theta(\det Dv)) (|Du - F|^p + |\theta(\det Du) - \theta(\det F)|) \circ v \, dx \leq c_\eta \delta, \quad (3.5)$$

where $v(x) = F^{-1}\varphi_\eta(x - a_0) + a_0$ and

$$c_\eta = 2^n \int_{B_{r/2}} (1 + \theta(\det F^{-1}D\varphi_\eta)) \, dx$$

(since W and F are fixed for the entire proof, we emphasize the dependence of the constants on η). For the rest of the proof we only need to deal with the fixed inner ball B' .

Let $R_\eta = \|Dv\|_{L^\infty}$, $M_\eta = \|D\varphi_\eta\|_{L^\infty}$. Since W is continuous in $\mathbb{R}_+^{n \times n}$ there is $\varepsilon \in (0, 1)$ such that

$$\begin{aligned} |W(\xi) - W(\zeta)| &\leq \eta \text{ for all } \xi, \zeta \in \mathbb{R}_+^{n \times n} \text{ with } |\zeta| \leq M_\eta, \\ \det \zeta &\geq \gamma \text{ and } |\xi - \zeta| \leq \varepsilon R_\eta, \end{aligned} \quad (3.6)$$

where γ was defined in (3.4) and depends only on W , F and η . Moreover, W^{qc} is continuous in $\mathbb{R}_+^{n \times n}$. This is proven for example in [22, Th. 2.4 and Prop. 2.3] by showing that W^{qc} is rank-one convex, and hence separately convex, in the open set $\mathbb{R}_+^{n \times n}$. Hence we may assume additionally that

$$|W^{\text{qc}}(\xi) - W^{\text{qc}}(F)| + |\theta(\det \xi) - \theta(\det F)| \leq \eta \text{ for all } \xi \text{ with } |\xi - F| \leq \varepsilon. \quad (3.7)$$

The parameter ε depends on η , but not on u and δ . We compute, with $\widehat{\varphi}_\eta(x) = \varphi_\eta(x - a_0)$,

$$\begin{aligned} \int_{B'} (W(Dz) - W^{\text{qc}}(Du)) \, dx &= \int_{B'} (W(Dz) - W(D\widehat{\varphi}_\eta)) \, dx \\ &\quad + \int_{B'} (W(D\widehat{\varphi}_\eta) - W^{\text{qc}}(F)) \, dx \\ &\quad + \int_{B'} (W^{\text{qc}}(F) - W^{\text{qc}}(Du)) \, dx \end{aligned} \quad (3.8)$$

and estimate the three terms separately.

The second term in (3.8) is bounded by $\eta|B'|$ by the definition of φ_η , see (3.3). To treat the third one we use (3.7) to obtain

$$W^{\text{qc}}(F) \leq W^{\text{qc}}(Du) + \eta \quad \text{on the set where } |Du - F| \leq \varepsilon.$$

The complement is small, indeed, from (3.1) we obtain

$$\begin{aligned} \int_{B'} (W^{\text{qc}}(F) - W^{\text{qc}}(Du)) dx &\leq \eta|B'| + W^{\text{qc}}(F) \mathcal{L}^n(|Du - F| > \varepsilon) \\ &\leq \eta|B'| + W^{\text{qc}}(F) \frac{1}{\varepsilon^p} |B| \delta. \end{aligned}$$

To estimate the first term in (3.8) we distinguish three subsets: $\omega = B' \cap \{|Du - F| \circ v \geq \varepsilon\}$, $\omega_d = B' \cap \{\det D\widehat{\varphi}_\eta < \gamma\} \setminus \omega$ and the rest $B' \setminus \omega \setminus \omega_d$.

In $B' \setminus \omega \setminus \omega_d$ we have $|Dv| \leq R_\eta$, $\det D\widehat{\varphi}_\eta \geq \gamma$, $|Du - F| \circ v \leq \varepsilon$ and therefore, since

$$Dz = Du \circ v Dv = (Du - F) \circ v Dv + D\widehat{\varphi}_\eta$$

we obtain

$$|Dz - D\widehat{\varphi}_\eta| \leq |Du - F| \circ v |Dv| \leq \varepsilon R_\eta.$$

By the continuity estimate (3.6) we obtain

$$|W(Dz) - W(D\widehat{\varphi}_\eta)| \leq \eta$$

and therefore

$$\int_{B' \setminus \omega_d \setminus \omega} (W(Dz) - W(D\widehat{\varphi}_\eta)) dx \leq \eta|B'|.$$

In the two error sets we use the growth estimate (2.1), which gives

$$W(Dz) \leq c(1 + |Du|^p \circ v |Dv|^p + \theta(\det Du \circ v \det Dv)). \quad (3.9)$$

With $|Dv| \leq R_\eta$ and (2.2) we obtain

$$W(Dz) \leq c(1 + R_\eta^p |Du|^p \circ v + (1 + \theta(\det Du) \circ v) (1 + \theta(\det Dv))), \quad (3.10)$$

where c only depends on W . At this point we treat the two error sets separately. For the estimate on ω we observe that $|Du - F| \geq \varepsilon$ implies

$$|Du| + 1 \leq |Du - F| + |F| + 1 \leq \left(\frac{|F| + 1}{\varepsilon} + 1 \right) |Du - F|$$

and

$$\theta(\det Du) \leq |\theta(\det Du) - \theta(\det F)| + \frac{\theta(\det F)}{\varepsilon^p} |Du - F|^p.$$

Therefore (3.10) gives

$$\begin{aligned} \int_{\omega} W(Dz) &\leq c \int_{\omega} (1 + \theta(\det Dv))(1 + R_{\eta}^p |Du|^p + \theta(\det Du)) \circ v \, dx \\ &\leq c_{\eta} \int_{\omega} (1 + \theta(\det Dv))(|Du - F|^p + |\theta(\det Du) - \theta(\det F)|) \circ v \, dx \\ &\leq c_{\eta} |B'| \delta. \end{aligned}$$

where in the last step we used (3.5). The constant depends on W , F and η (via ε), but not on δ and u .

In ω_d instead we have $|Du - F| \circ v \leq \varepsilon$. Then, recalling the continuity estimate (3.7), we have $|Du| \circ v \leq |F| + 1$ and $\theta(\det Du \circ v) \leq \theta(\det F) + 1$ and therefore (3.10) reduces to

$$\begin{aligned} W(Dz) &\leq c(1 + R_{\eta}^p(1 + |F|^p) + (1 + \theta(\det F))(1 + \theta(\det Dv))) \\ &\leq c_*(1 + \|F^{-1} D\varphi_{\eta}\|_{\infty}^p)(1 + |F|^p + \theta(\det F))(1 + \theta(\det Dv)), \quad (3.11) \end{aligned}$$

with a constant $c_* > 0$ which depends only on W . With (3.4) we conclude

$$\int_{\omega_d} W(Dz) dx \leq c_* |B'| \eta.$$

Adding all terms we obtain

$$\int_{B'} (W(Dz) - W^{\text{qc}}(Du)) \, dx \leq (3\eta + 2^n \frac{W^{\text{qc}}(F)}{\varepsilon^p} \delta + c_{\eta} \delta + c_* \eta) |B'|.$$

Since c_* depends only on W , choosing δ sufficiently small the proof is concluded. \square

3.2 Upper bound

Lemma 3.3 (Recovery sequence). *Let $\Omega \subset \mathbb{R}^n$ open, Lipschitz, bounded, let $W \in C^0(\mathbb{R}_+^{n \times n}, [0, \infty))$ obey (2.1–2.2). Then for any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $\det Du > 0$ almost everywhere there is a sequence $u_j \rightharpoonup u$ in $W^{1,p}$ such that $\det Du_j > 0$ almost everywhere, $u_j = u$ on $\partial\Omega$ and*

$$\limsup_{j \rightarrow \infty} \int_{\Omega} W(Du_j) dx \leq \int_{\Omega} W^{\text{qc}}(Du) dx.$$

If additionally $u \in W^{1,\infty}$ then also $u_j \in W^{1,\infty}$.

Proof. Fix $\eta > 0$. It suffices to construct w with $\|u - w\|_p \leq \eta$, $w = u$ on $\partial\Omega$, and $\int_{\Omega} W(Dw) dx \leq \int_{\Omega} W^{\text{qc}}(Du) dx + \eta$.

If $\int_{\Omega} W^{\text{qc}}(Du) dx = \infty$ the constant sequence will do, hence we can assume that $W^{\text{qc}} \circ Du \in L^1$. By convexity of θ and of the p -norm the definition of $W^{\text{qc}}(F)$ gives

$$\frac{1}{c} |F|^p + \frac{1}{c} \theta(\det F) - c \leq W^{\text{qc}}(F) \text{ for all } F,$$

therefore $|Du|^p$ and $\theta(\det Du)$ are also integrable. We denote by E the set of Lebesgue points of Du and $\theta(\det Du)$. For every $x \in E$ we set $F(x) = Du(x)$ and choose $\delta(x)$ as in Lemma 3.2 for this F and η as above.

The construction is done by successive application of Lemma 3.2. We set $w_0 = u$, $\Omega_0 = \Omega$ and describe how to pass from (w_j, Ω_j) to (w_{j+1}, Ω_{j+1}) . For all $x \in E \cap \Omega_j$ we choose $r_j(x) \in (0, \eta)$ such that $B(x, r_j(x)) \subset \Omega_j$ and

$$\int_{B(x, r)} (|Dw_j - F(x)|^p + |\theta(\det Dw_j) - \theta(\det F(x))|) dx' \leq \delta(x)$$

for all $r < r_j(x)$. This gives a fine cover of $E \cap \Omega_j$. We extract a disjoint subcover $B(x_k, r_k)_{k \in \mathbb{N}}$ and from this subcover finitely many balls $B(x_k, r_k)_{k=0, \dots, M}$ which cover at least half the volume of Ω_j .

We set $w_{j+1} = w_j$ on $\Omega \setminus \cup_{k=0}^M B(x_k, r_k)$ and define w_{j+1} as the result of Lemma 3.2 in each of the balls. Then $w_{j+1} \in W^{1,p}(\Omega; \mathbb{R}^n)$ and $w_{j+1} = w_j = u$ on $\partial\Omega$. Further, the smaller balls $B(x'_k, r_k/2) \subset B(x_k, r_k)$ obey

$$\int_{B(x'_k, r_k/2)} W(Dw_{j+1}) dx \leq \int_{B(x'_k, r_k/2)} (W^{\text{qc}}(Du) + \eta) dx \quad (3.12)$$

and

$$\int_{B(x'_k, r_k/2)} |w_{j+1} - u|^p dx \leq c\eta^p \int_{B(x'_k, r_k/2)} (1 + W^{\text{qc}}(Du)) dx, \quad (3.13)$$

with $w_{j+1} = w_j$ outside these balls. Finally we set $\Omega_{j+1} = \Omega_j \setminus \cup_{k=0}^M \overline{B}(x'_k, r_k/2)$, so that $|\Omega_{j+1}| \leq (1 - 2^{-n-1})|\Omega_j|$, and iterate. We remark that $w_{j+1} = u$ on the open set Ω_{j+1} , hence there is no need to redefine E , F and δ at each step. This concludes the construction of the sequence w_j .

It remains to show that w_j , for j sufficiently large, has the desired properties. Each of these functions coincides with u outside a finite number of disjoint balls, and has been modified exactly once in each of those balls. By (3.13) we have

$$\int_{\Omega} |w_j - u|^p dx \leq c\eta^p \int_{\Omega} (1 + W^{\text{qc}}(Du)) dx$$

hence w_j is close to u in L^p , independently of j .

Analogously from (3.12) we deduce, for the union of the balls $\Omega \setminus \Omega_j$,

$$\int_{\Omega \setminus \Omega_j} W(Dw_j) dx \leq \int_{\Omega \setminus \Omega_j} (W^{\text{qc}}(Du) + \eta) dx$$

which implies

$$\int_{\Omega} W(Dw_j) dx \leq \int_{\Omega \setminus \Omega_j} (W^{\text{qc}}(Du) + \eta) dx + \int_{\Omega_j} W(Du) dx.$$

Since $|\Omega_j| \leq (1 - 2^{-n-1})^j |\Omega| \rightarrow 0$ and by the growth condition $W(Du) \in L^1(\Omega)$, for sufficiently large j we have

$$\int_{\Omega} W(Dw_j) dx \leq \int_{\Omega} (W^{\text{qc}}(Du) + 2\eta) dx,$$

as required. \square

Lemma 3.4 (Quasiconvexity). *The function W^{qc} is quasiconvex.*

Remark 3.5. W^{qc} is the largest (extended-valued) quasiconvex function below W , hence in this sense its quasiconvex envelope. This function does not necessarily coincide with the supremum of all finite-valued quasiconvex functions below W ; in particular, this is not true for the function discussed in [8, Example 3.5].

Proof. Fix F with $\det F > 0$, $\Omega = B_1$, $\psi \in W^{1,\infty}(B_1, \mathbb{R}^n)$ with $\psi(x) = Fx$ on ∂B_1 . We need to show that

$$W^{\text{qc}}(F) \leq \int_{B_1} W^{\text{qc}}(D\psi) dx.$$

By Lemma 3.3 there is a sequence $\varphi_j \in W^{1,\infty}$ with $\varphi_j(x) = \psi(x) = Fx$ on ∂B_1 and such that

$$\limsup_{j \rightarrow \infty} \int_{B_1} W(D\varphi_j) dx \leq \int_{B_1} W^{\text{qc}}(D\psi) dx.$$

Since every φ_j is admissible in the definition of $W^{\text{qc}}(F)$ we obtain

$$W^{\text{qc}}(F) \leq \int_{B_1} W(D\varphi_j) dx$$

for all j , and in particular

$$W^{\text{qc}}(F) \leq \int_{B_1} W^{\text{qc}}(D\psi) dx$$

as desired. \square

4 Construction of volume-preserving maps

In this case the translation is not needed, and correspondingly the proof of Lemma 4.2 is simpler than the one of Lemma 3.2; we give it in detail since it illustrates in a compact way the key ideas of our construction. At the same time the continuity of W^{qc} is less clear than for orientation-preserving maps. It essentially follows from the results of [37, 13]. Since it was not stated there we briefly show how it can be derived from the construction in [13].

Lemma 4.1. *Given $W : \Sigma \rightarrow [0, \infty)$, extended by ∞ elsewhere, we define W^{qc} by (1.3). The function W^{qc} is rank-one convex and hence continuous on Σ .*

Proof. We first observe that, by general scaling and covering arguments, the definition of W^{qc} does not depend on the domain, and in particular

$$W^{\text{qc}}(F) = \inf \left\{ \int_{\omega} W(D\varphi) dx : \varphi \in W^{1,\infty}(\omega; \mathbb{R}^n), \varphi(x) = Fx \text{ for } x \in \partial\omega \right\} \quad (4.1)$$

for any bounded open nonempty polyhedron $\omega \subset \mathbb{R}^n$.

To prove rank-one convexity we fix $A, B \in \Sigma$ with $\text{rank}(A - B) = 1$ and $\lambda \in (0, 1)$. We define $F = \lambda A + (1 - \lambda)B$. By the construction in [13, Th. 2.1] (with $n = m = r$, $P = Q = \text{Id}$, $t = 1$, $\varepsilon = 1$) there is a finite set $K \subset \Sigma$ such that for any $\delta > 0$ one can find a polyhedron Ω and a piecewise affine function $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that $u(x) = Fx$ on $\partial\Omega$, $Du \in K \subset \Sigma$ almost everywhere, $|\{Du \notin \{A, B\}\}| \leq \delta|\Omega|$. The latter, together with the boundary data, implies

$$|\{Du = A\}| \leq (\lambda + c\delta)|\Omega| \text{ and } |\{Du = B\}| \leq (1 - \lambda + c\delta)|\Omega|,$$

with c depending on A and B . Further, the set $\{Du = A\}$ is a finite union of simplexes ω_j^A . For each of them there is, by (4.1) with $F = A$, a Lipschitz function v_j^A with $v_j^A = u$ on $\partial\omega_j^A$ and

$$\int_{\omega_j^A} W(Dv_j^A) dx \leq |\omega_j^A| (W^{\text{qc}}(A) + \delta).$$

The same holds for the set $\{Du = B\}$. We set $w = v_j^A$ on each ω_j^A , $w = v_j^B$ on each ω_j^B , $w = u$ on the rest. Since $w(x) = Fx$ on $\partial\Omega$ we have

$$\begin{aligned} W^{\text{qc}}(F) &\leq \int_{\Omega} W(Dw) dx \leq \frac{|\{Du = A\}|}{|\Omega|} (W^{\text{qc}}(A) + \delta) \\ &\quad + \frac{|\{Du = B\}|}{|\Omega|} (W^{\text{qc}}(B) + \delta) + \frac{|\{Du \notin \{A, B\}\}|}{|\Omega|} \max W(K) \\ &\leq \lambda W^{\text{qc}}(A) + (1 - \lambda) W^{\text{qc}}(B) + c\delta \max W(K), \end{aligned}$$

with c depending on A and B . Taking $\delta \rightarrow 0$ (with fixed K) this implies the desired inequality $W^{\text{qc}}(F) \leq \lambda W^{\text{qc}}(A) + (1 - \lambda) W^{\text{qc}}(B)$. Since W^{qc} is rank-one convex, it is separately convex in suitable variables and hence continuous (for details see, e.g., [13, Step 2 in the proof of Th. 3.1]). \square

Lemma 4.2. *Let $W \in C^0(\Sigma; [0, \infty))$ obey (2.5) for some $p \geq 1$. Then for any $F \in \mathbb{R}^{n \times n}$ and $\eta > 0$ there is $\delta > 0$ such that the following holds: For any ball $B = B(x_0, r)$ and any function $u \in W^{1,p}(B, \mathbb{R}^n)$ with*

$$\int_B |Du - F|^p dx \leq \delta \text{ and } Du \in \Sigma \text{ a.e.}$$

one can find $z \in W^{1,p}(B, \mathbb{R}^n)$ with $u = z$ on ∂B ,

$$\int_B W(Dz)dx \leq \int_B (W^{\text{qc}}(Du) + \eta)dx \text{ and } Dz \in \Sigma \text{ a.e.}$$

Additionally,

$$\int_B |u - z|^p dx \leq cr^p \int_B (1 + W^{\text{qc}}(Du))dx.$$

Proof. Let $\varphi_\eta \in W^{1,\infty}(B, \mathbb{R}^n)$ be such that $\varphi_\eta(x) = Fx$ on ∂B and

$$\int_B W(D\varphi_\eta)dx \leq W^{\text{qc}}(F) + \eta.$$

We define

$$v = F^{-1}\varphi_\eta$$

and observe that, by [3, Theorem 2], v is a bilipschitz map from B onto itself. Therefore we can define

$$z = u \circ v \in W^{1,p}(B, \mathbb{R}^n)$$

and compute its gradient

$$Dz = Du \circ v Dv = (Du - F) \circ v Dv + D\varphi_\eta.$$

We set $R_\eta = \|Dv\|_\infty$, $M_\eta = \|D\varphi_\eta\|_\infty$ and choose $\varepsilon \in (0, 1)$ such that

$$|W(\xi) - W(\zeta)| \leq \eta \text{ whenever } |\zeta| \leq M_\eta \text{ and } |\xi - \zeta| \leq \varepsilon R_\eta \quad (4.2)$$

and

$$|W^{\text{qc}}(\xi) - W^{\text{qc}}(F)| \leq \eta \text{ whenever } |\xi - F| \leq \varepsilon. \quad (4.3)$$

In order to estimate the integral

$$\begin{aligned} \int_B (W(Dz) - W^{\text{qc}}(Du))dx &= \int_B (W(Dz) - W(D\varphi_\eta))dx \\ &\quad + \int_B (W(D\varphi_\eta) - W^{\text{qc}}(F))dx \\ &\quad + \int_B (W^{\text{qc}}(F) - W^{\text{qc}}(Du))dx \end{aligned} \quad (4.4)$$

we consider the three terms separately. The second integral in (4.4) is bounded by $\eta|B|$ by the definition of φ_η . In order to estimate the last integral in (4.4) we use (4.3) to obtain

$$W^{\text{qc}}(F) \leq W^{\text{qc}}(Du) + \eta \quad \text{on the set where } |Du - F| \leq \varepsilon.$$

The complement is small and gives a small contribution. Precisely,

$$\begin{aligned} \int_B (W^{\text{qc}}(F) - W^{\text{qc}}(Du)) dx &\leq \eta |B| + W^{\text{qc}}(F) \mathcal{L}^n(|Du - F| > \varepsilon) \\ &\leq \eta |B| + W^{\text{qc}}(F) \frac{1}{\varepsilon^p} |B| \delta. \end{aligned}$$

In order to estimate the first integral in (4.4) we distinguish the set $\omega = \{|Du - F| \circ v > \varepsilon\}$ and the rest. On $B \setminus \omega$, from the explicit expression for Dz we obtain

$$|Dz - D\varphi_\eta| \leq \|Dv\|_\infty |Du - F| \circ v \leq \varepsilon R_\eta$$

and recalling (4.2) we can estimate

$$|W(Dz) - W(D\varphi_\eta)| \leq \eta \text{ on } B \setminus \omega.$$

Since $|Du - F| \geq \varepsilon$ implies

$$|Du| + 1 \leq |Du - F| + |F| + 1 \leq \left(\frac{|F| + 1}{\varepsilon} + 1 \right) |Du - F|,$$

the contribution of ω can be estimated by

$$\int_\omega W(Dz) dx \leq c \int_\omega (R_\eta^p |Du|^p \circ v + 1) dx \leq c_{F,\varepsilon} \int_\omega |Du - F|^p \circ v dx$$

where the constant depends on F , η and ε . Finally, v is a bilipschitz map from B onto itself with $\det Dv = 1$ almost everywhere (see [3, Theorem 2]) and therefore

$$\int_\omega W(Dz) dx \leq c_{F,\varepsilon} \int_B |Du - F|^p \circ v dx = c_{F,\varepsilon} \int_B |Du - F|^p dx \leq c_{F,\varepsilon} \delta |B|.$$

Collecting terms we conclude

$$\oint_B (W(Dz) - W^{\text{qc}}(Du)) dx \leq 3\eta + \frac{W^{\text{qc}}(F)}{\varepsilon^p} \delta + c_{F,\varepsilon} \delta.$$

Since ε depends on η and F but not on δ and u , choosing δ sufficiently small the proof is concluded. The L^p estimate follows from the growth estimate and Poincaré's inequality. \square

Lemma 4.3 (Recovery sequence). *Let $\Omega \subset \mathbb{R}^n$ open, Lipschitz, bounded, $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $Du \in \Sigma$ almost everywhere. Then there is a sequence $u_j \rightharpoonup u$ in $W^{1,p}$ such that $\det Du_j \in \Sigma$ almost everywhere and*

$$\limsup_{j \rightarrow \infty} \int W(Du_j) dx \leq \int W^{\text{qc}}(Du) dx.$$

If additionally $u \in W^{1,\infty}$ then also $u_j \in W^{1,\infty}$.

Proof. The proof is just like the one in Lemma 3.3, for brevity we do not repeat it. \square

Lemma 4.4 (Quasiconvexity). *The function W^{qc} is quasiconvex.*

Proof. The proof is just like the one of Lemma 3.4, for brevity we do not repeat it. \square

A Composition of Sobolev functions with Lipschitz functions

The composition of a Lipschitz with a Sobolev function and the composition of a Sobolev with a bilipschitz function are standard. Although there is a substantial literature on the subject, see for example [24, 30] and references therein, we have been unable to find the statement needed here on the composition of a Sobolev with a Lipschitz function, hence we give a short self-contained proof. To see the difficulty with measurability one can consider the example $\psi(x_1, x_2) = (x_1, 0)$, $f(x_1, x_2) = h(x_1)\chi_{\{0\}}(x_2)$, with $h : \mathbb{R} \rightarrow \mathbb{R}$ not measurable. Then $f = 0$ \mathcal{L}^2 -almost everywhere but $f \circ \psi$ is not measurable. To see the difficulty with integrability one can consider $f(x) = |x|^{-1/2}$ in the unit ball of \mathbb{R}^2 , with $\psi(x) = |x|x$ around the origin. Then f is in $W^{1,1}$ but $(f \circ \psi)(x) = |x|^{-1}$ is not.

Lemma A.1. *Let $\psi \in W^{1,\infty}(B_1; \overline{B}_1)$, $f_k \in L^1(B_2; \mathbb{R}^m)$ with $\sum_k \|f_k\|_{L^1(B_2)} < \infty$. Then the maps $(x, a_0) \mapsto f_k(a_0 + \psi(x - a_0))$ are \mathcal{L}^{2n} measurable and for almost all $a_0 \in B_1$ the functions*

$$z_k(x) = f_k(a_0 + \psi(x - a_0))$$

are in $L^1(B(a_0, 1))$ with $\sum_k \|z_k\|_{L^1(B(a_0, 1))} < \infty$.

Proof. We define the continuous function $g : B_1 \times B_1 \rightarrow B_2$ by $g(x, y) = x + \psi(y)$ and show that for any k the function $f_k \circ g$ is \mathcal{L}^{2n} -measurable.

Let $A \subset \mathbb{R}^m$ be open. Then $f_k^{-1}(A) \subset B_2$ is \mathcal{L}^n -measurable, therefore $f_k^{-1}(A) = E \setminus N$, with E Borel and N a null set. Since g is continuous, $g^{-1}(E)$ is Borel. It remains to show that $|N| = 0$ implies $g^{-1}(N) = 0$. Let $F \subset B_2$ be Borel with $N \subset F$ and $|F| = 0$. Then $g^{-1}(F)$ is Borel and \mathcal{L}^{2n} -measurable. For any $y \in \mathbb{R}^n$, the set $T_y = \{x \in \mathbb{R}^n : g(x, y) \in F\} = F - \psi(y)$ is a \mathcal{L}^n -null set. By Fubini's theorem

$$\mathcal{L}^{2n}(g^{-1}(F)) = \int_{\mathbb{R}^n} \mathcal{L}^n(T_y) dy = 0.$$

Therefore $f_k \circ g$ is measurable. A second application of Fubini's theorem shows that for almost all $a_0 \in B_1$ each function $x \mapsto f_k(g(a_0, x))$ is measurable;

clearly the same holds for the translations $z_k(x) = f_k(g(a_0, x - a_0))$. We conclude that for almost all $a_0 \in B_1$ all the functions z_k are \mathcal{L}^n -measurable.

We define $A : B_1 \rightarrow [0, \infty]$ by

$$\begin{aligned} A(a_0) &= \sum_{k \in \mathbb{N}} \|z_k\|_{L^1(B(a_0, 1))} = \sum_{k \in \mathbb{N}} \int_{B(a_0, 1)} |f_k|(a_0 + \psi(x - a_0)) dx \\ &= \sum_{k \in \mathbb{N}} \int_{B_1} |f_k|(a_0 + \psi(x')) dx'. \end{aligned}$$

The integrand is nonnegative and measurable, hence we can interchange the order of summation and integration. Since the integrand is measurable as a function on \mathbb{R}^{2n} , the function A is measurable. Integrating and changing variables as usual,

$$\int_{B_1} A(a_0) da_0 \leq \sum_{k \in \mathbb{N}} \int_{B_1} \|f_k\|_{L^1(B_2)} dx' \leq |B_1| \sum_{k \in \mathbb{N}} \|f_k\|_{L^1(B_2)} < \infty.$$

Therefore $A(a_0) < \infty$ almost everywhere, which concludes the proof. \square

Lemma A.2 (Chain rule). *Let $\psi \in W^{1, \infty}(B_1; \overline{B_1})$, $u \in W^{1, 1}(B_2; \mathbb{R}^m)$. Then for almost all $a_0 \in B_1$ the function $w(x) = u(a_0 + \psi(x - a_0))$ belongs to $W^{1, 1}(B(a_0, 1); \mathbb{R}^m)$ with*

$$Dw(x) = Du(a_0 + \psi(x - a_0))D\psi(x - a_0).$$

If $\psi(x) = x$ on ∂B_1 then $w = u$ (as traces) on $\partial B(a_0, 1)$.

Proof. We choose a sequence $u_k \in C^\infty(\overline{B_2}; \mathbb{R}^m)$ such that $\|u_k - u\|_{W^{1, 1}(B_2)} \leq 2^{-k}$ and apply Lemma A.1 to the sequence $f_k = (u_k - u, Du_k - Du) \in L^1(B_2; \mathbb{R}^m \times \mathbb{R}^{m \times n})$, which obeys $\sum \|f_k\|_{L^1} \leq 2$. For any fixed a_0 not in the null set given by the lemma, we obtain the corresponding sequence z_k with the properties asserted in Lemma A.1. Additionally we define w_k by $w_k(x) = u_k(a_0 + \psi(x - a_0))$ and w as in the statement. Each of the functions z_k with values in $\mathbb{R}^m \times \mathbb{R}^{m \times n}$ is measurable, therefore the first m components which are given by $w_k - w$ are measurable. The continuity of w_k implies the measurability of w . Furthermore,

$$\begin{aligned} \|w_k - w\|_{L^1(B(a_0, 1))} &= \int_{B(a_0, 1)} |u_k - u|(a_0 + \psi(x - a_0)) dx \\ &\leq \int_{B(a_0, 1)} |f_k|(a_0 + \psi(x - a_0)) dx = \|z_k\|_{L^1(B(a_0, 1))} \rightarrow 0. \end{aligned}$$

We conclude $w \in L^1$ and $w_k \rightarrow w$ in L^1 .

We now repeat the procedure for the gradient. We denote by $F(x) = Du(a_0 + \psi(x - a_0))D\psi(x - a_0)$ the expression given in the statement. Since every u_k is smooth by the usual chain rule we obtain

$$Dw_k(x) = Du_k(a_0 + \psi(x - a_0))D\psi(x - a_0),$$

which is the product of a continuous and an L^∞ function and therefore measurable. Further,

$$(Dw_k - F)(x) = (Du_k - Du)(a_0 + \psi(x - a_0))D\psi(x - a_0).$$

The first factor is the second component of z_k hence measurable by Lemma A.1, the second belongs to L^∞ . Continuity of Dw_k gives measurability of F . Further,

$$\begin{aligned} \|Dw_k - F\|_{L^1(B(a_0,1))} &\leq \|D\psi\|_\infty \int_{B(a_0,1)} |Du_k - Du|(a_0 + \psi(x - a_0))dx \\ &\leq \|D\psi\|_\infty \int_{B(a_0,1)} |f_k|(a_0 + \psi(x - a_0))dx \rightarrow 0. \end{aligned}$$

Therefore $F \in L^1$ and $Dw_k \rightarrow F$ in L^1 . Continuity of the distributional derivative implies $F = Dw$ distributionally and $w \in W^{1,1}$.

To obtain the condition on the trace it suffices to extend ψ to be the identity outside B_1 , u to a function in $W^{1,1}(\mathbb{R}^n; \mathbb{R}^m)$ and work on a larger ball. \square

B Construction for submultiplicative integrands

We show here how our construction of the recovery sequence can be extended to the more general situation discussed in Remark 2.3. We focus on the orientation-preserving case, the other one is simpler. For brevity we only show how the basic construction step is modified, the covering of Lemma 3.3 is not significantly changed. Indeed, it suffices to use $p = 1$ and takes Lebesgue points of Du and $W(Du)$ instead of Lebesgue points of Du and $\theta(\det Du)$; $W(Du) \in L^1$ by the growth condition (2.4).

Lemma B.1. *Assume that $W \in C^0(\mathbb{R}_+^{n \times n}, [0, \infty))$ satisfies*

$$\frac{1}{c}|F| - c \leq W(G) \tag{B.1}$$

and

$$W(FG) \leq c_W(1 + W(F))(1 + W(G)) \tag{B.2}$$

for all $F, G \in \mathbb{R}_+^{n \times n}$, with a fixed $c_W > 0$. Then for any $F \in \mathbb{R}_+^{n \times n}$ and $\eta > 0$ there is $\delta > 0$ such that for any $B = B(x_0, r)$ and $u \in W^{1,1}(B, \mathbb{R}^n)$ with

$$\int_B (|Du - F| + |W(Du) - W(F)|) dx \leq \delta \text{ and } \det Du > 0 \text{ a.e.}$$

there are $a_0 \in B(x_0, r/2)$ and $z \in W^{1,1}(B, \mathbb{R}^n)$ with $\det Dz > 0$ a.e., $z = u$ on $B(x_0, r) \setminus B(a_0, r/2)$ and

$$\int_{B(a_0, r/2)} W(Dz) dx \leq \int_{B(a_0, r/2)} (W^{\text{qc}}(Du) + \eta) dx.$$

Additionally,

$$\int_B |u - z| dx \leq cr \int_B (W^{\text{qc}}(Du) + 1) dx.$$

If u is Lipschitz, then so is z .

Proof. This is very similar to the proof of Lemma 3.2, we only discuss the differences. After (3.3), (B.2) implies $W(F^{-1}D\varphi_\eta) \in L^1$ and equation (3.4) is replaced by

$$\int_{B_{r/2} \cap \{\det D\varphi_\eta < \gamma\}} (1 + W(F^{-1}D\varphi_\eta)) dx \leq \frac{|B_{r/2}|}{c_W(2 + W(F))} \eta. \quad (\text{B.3})$$

In Lemma 3.1 we use $f = |Du - F| + |W(Du) - W(F)|$ and $g = 1 + W(F^{-1}D\varphi_\eta)$, (3.5) is replaced by

$$\int_{B'} (1 + W(F^{-1}D\widehat{\varphi}_\eta)) (|Du - F| + |W(Du) - W(F)|) \circ v dx \leq c_\eta \delta, \quad (\text{B.4})$$

where $c_\eta = 2^n \int_{B_{r/2}} (1 + W(F^{-1}D\varphi_\eta)) dx < \infty$. In (3.7) we use continuity of W instead of θ . The remaining differences are in the treatment of the two error sets. We replace (3.9) by

$$W(Dz) \leq c_W(1 + W(Du) \circ v)(1 + W(Dv)). \quad (\text{B.5})$$

We start from ω . From $|Du - F| \circ v \geq \varepsilon$ we deduce

$$\begin{aligned} 1 + W(Du) \circ v &\leq 1 + W(F) + |W(Du) - W(F)| \circ v \\ &\leq c_{F,\varepsilon} (|Du - F| + |W(Du) - W(F)|) \circ v \end{aligned}$$

where $c_{F,\varepsilon} = 1 + (1 + W(F))/\varepsilon$. Therefore the estimate (B.5) gives

$$\int_\omega W(Dz) dx \leq c_W c_{F,\varepsilon} \int_{B'} (1 + W(Dv)) (|Du - F| + |W(Du) - W(F)|) \circ v dx$$

Recalling (B.4), which had been obtained by the choice of a_0 , we get

$$\int_\omega W(Dz) dx \leq c_W c_{F,\varepsilon} c_\eta \delta.$$

The constant depends on ε and F (and hence on η) but not on δ and u .

In ω_d instead we have $|Du - F| \circ v \leq \varepsilon$. The continuity estimate (3.7) gives then $W(Du \circ v) \leq W(F) + 1$ and therefore (B.5) reduces to

$$W(Dz) \leq c_W(2 + W(F))(1 + W(Dv)).$$

With (B.3) we conclude

$$\int_{\omega_d} W(Dz) dx \leq |B'| \eta.$$

The conclusion is the same. \square

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